International Journal of Theoretical Physics, Vol. 31, No. 2, 1992

Uniform Stability with Respect to the Impulsive Perturbations of the Solutions of Impulsive Differential Equations

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Received October 2, 1990

The notions of strong uniform stability of the zero solution of systems of differential equations and uniform stability with respect to the impulsive perturbations of the zero solution of systems of impulsive differential equations are introduced. The main results are given in two theorems. The first contains sufficient conditions under which the strong uniform stability of the zero solution of the respective system without impulses implies uniform stability with respect to the impulsive perturbations of the zero solution of the initial system with impulses. In the second theorem sufficient conditions are given under which the uniform Lipschitz stability of the zero solution of the respective system without impulses implies uniform stability with respect to the impulsive perturbations of the zero solution of the initial system with impulses.

1. INTRODUCTION

Numerous evolutionary processes during their development are subject to short-time perturbations. In many cases the duration of these perturbations is comparatively small and we can assume that they are realized momentarily in the form of impulses. An adequate mathematical model of such processes is given by the systems of ordinary impulsive differential equations which are the object of investigation of the present paper. Because of the complexity of these systems, it is possible to find their solutions in a closed form only in exceptional cases. That is why it is appropriate to develop their qualitative theory. Many papers (e.g., Mil'man and Myshkis, 1960; Pandit, 1977; Rao and Rao, 1977; Simeonov and Bainov, 1986) have proposed specific criteria for stability of the solutions of systems of ordinary

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0020-7748/92/0200-0363$06.50/0

impulsive differential equations. The definitions of stability introduced in the cited papers represent a modification of the respective classical definitions. In this paper a type of stability characteristic only for solutions of this type of system is introduced, namely uniform stability with respect to the impulsive perturbations.

2. STATEMENT OF THE PROBLEM

Consider the following initial value problem for systems of impulsive differential equations:

$$\frac{dx}{dt}=f(t, x), \qquad t\neq\tau_i \tag{1}$$

$$\Delta x(t)|_{t=\tau_i}=I_{j_i}(x(\tau_i)) \tag{2}$$

$$x(\tau_0)=x_0 \tag{3}$$

where f: $R^+\times D\to R^n$, $R^+=[0, +\infty)$, D is a domain in R^n; I_i: $D\to R^n$, $i=1, 2, \ldots$; $(\tau_0, x_0)\in R^+\times D$; and

$$\Delta x(t)|_{t=\tau_i}=x(\tau_i+0)-x(\tau_i)$$

At the moments $\tau_1, \tau_2, \ldots$ $(\tau_1<\tau_2<\cdots)$ the integral curve of problem (1)–(3) meets some of the hypersurfaces

$$\sigma_i:\quad t=t_i(x), \quad i=1, 2, \ldots \tag{4}$$

where t_i: $D\to R^+$. In equality (2) by j_i we have denoted the number of the hypersurface met by the integral curve of the problem considered at the moment τ_i, $i=1, 2, \ldots$. We note that in general $i\neq j_i$ [see Example 1 of Dishliev and Bainov (1985)]. The solution of problem (1)–(3) is a piecewise continuous function with points of discontinuity $\tau_1, \tau_2, \ldots$ of the first kind at which it is continuous from the left. In more detail, the solution of problem (1)–(3) is defined as follows:

(i) For $\tau_0\leq t\leq\tau_1$ it coincides with the solution of problem (1), (3).

(ii) For $\tau_i<t\leq\tau_{i+1}$ it coincides with the solution of system (1) with initial condition

$$x(\tau_i+0)=x(\tau_i)+I_{j_i}(x(\tau_i)), \qquad i=1, 2, \ldots$$

If after an impulse the integral curve again hits a hypersurface of (4), i.e., if $\tau_i=t_k(x(\tau_i)+I_{j_i}(x(\tau_i)))$, $i, k\in\{1, 2, \ldots\}$, then a new impulse at the moment τ_i is not realized.

Introduce the following notation: $x_I(t;\, \tau_0, x_0)$ is the solution of problem (1)–(3). The index I shows that the magnitudes of the impulses are determined by means of the set of functions $I=\{I_i, i=1, 2, \ldots\}$. Denote the solution of the respective initial value problem without impulses (1), (3) by $x(t;\tau_0,x_0)$. Introduce the sets $\Omega_i=\{(t,x);\, t_{i-1}(x)\le t<t_i(x),\, x\in D\}$, $i=1,2,\ldots$, $t_0(x)=0$ for $x\in D$. By $\|\cdot\|$ we denote the Euclidean norm in R^n. Introduce also the following sets: $B_\delta=\{x;\,\|x\|<\delta\}$, $\delta>0$; if the set $A\ne\varnothing$, then $B_\delta(A)=\{x;\,\|x-a\|<\delta,\, a\in A\}$, $B_\delta(\varnothing)=\varnothing$; ∂A is the set of all boundary points of A; $C[X,Y]$ is the set of all continuous functions $\varphi: X\to Y$.

3. PRELIMINARY NOTES

We say that conditions (A) are satisfied if the following conditions hold:

A1. The function $f\in[R^+\times D, R^n]$ and is locally Lipschitz continuous with respect to x in D with a constant independent of t.

A2. There exists a constant $M>0$ such that $\|f(t,x)\|\le M$ for $(t,x)\in R^+\times D$.

A3. For any point $(\tau_0,x_0)\in R^+\times D$ the solution of the problem without impulses (1), (3) does not leave the set D for $t\ge\tau_0$.

A4. The functions t_i are Lipschitz continuous with respect to x in D with respective constants $L_i<1/M$, $i=1,2,\ldots$.

A5. $0<t_1(x)<t_2(x)<\cdots$, $x\in D$.

Lemma 1 (Dishliev and Bainov, 1988). Let conditions (A) hold. Then, if $(\tau_0,x_0)\in\Omega_i$, the first hypersurface met by the integral curve $(t,x(t;\tau_0,x_0))$ for $t>\tau_0$ is σ_i.

Definition 1. We say that the solution $x_I(t;\tau_0,x_0)$ of problem (1)–(3) is quasiunique if in any of the intervals $(\tau_{i-1},\tau_i]$, $i=1,2,\ldots$, the solution of the problem without impulses

$$\frac{dx}{dt}=f(t,x),\qquad x(\tau_{i-1})=x_I(\tau_{i-1};\tau_0,x_0)+I_{j_{i-1}}(x_I(\tau_{i-1};\tau_0,x_0))$$

is unique.

We shall note that by Definition 1, after an impulse it is possible for different integral curves to merge. This is due to the fact that some of the functions $(E+I_i)$ may not be bijective. Here E is the identity in R^n.

By $x_{I^*}(t;\tau_0,x_0)$ we denote the solution of the following system with impulses:

$$\frac{dx}{dt}=f(t,x),\qquad t\ne\tau_i^* \tag{5}$$

$$\Delta x(t)|_{t=\tau_i^*}=I_{s_i}^*(x(\tau_i^*)),\qquad i=1,2,\ldots \tag{6}$$

with initial condition (3), where $I_i^*: D \to R^n$; $\tau_1^*, \tau_2^*, \dots$ $(\tau_1^* < \tau_2^* < \cdots)$ are the moments at which the integral curve $(t, x_{I^*}(t; \tau_0, x_0))$ meets the hypersurfaces (4) and s_i is the number of the hypersurface met by this integral curve at the moment τ_i^*, $i = 1, 2, \dots$. Note that in general $s_i \neq j_i$, $i = 2, 3, \dots$, i.e., the hypersurfaces met successively by the integral curves $(t, x_{I^*}(t; \tau_0, x_0))$ and $(t, x_I(t; \tau_0, x_0))$ may not coincide. We shall use the notations $x_i^* = x_{I^*}(\tau_i^*; \tau_0, x_0)$ and $x_i^{*+} = x_i^* + I_{s_i}^*(x_i^*)$, $i = 1, 2, \dots$.

We say that conditions (B) are satisfied if the following conditions hold:

B1. There exists a constant $\Delta > 0$ such that $(E + I_i): D \to D \backslash B_\Delta(\partial D)$, $i = 1, 2, \dots$.

B2. $t_i(x) \to \infty$ as $i \to \infty$, uniformly with respect to $x \in D$.

B3. There exists a constant $d > 0$ such that for any point $x \in D$ and $i = 1, 2, \dots$ we have $t_i(x) > t_i(x + I_i(x)) + d$.

Lemma 2. Let the following conditions be satisfied:

1. Conditions (A) and (B) hold.
2. $\|I_i^*(x) - I_i(x)\| < \min(\Delta, Md)$, $x \in D$, $i = 1, 2, \dots$.

Then for any point $(\tau_0, x_0) \in R^+ \times D$ the solution $x_{I^*}(t; \tau_0, x_0)$ of problem (5), (6), (3) is quasiunique and continuable for any $t \geq \tau_0$.

Proof. Let $(\tau_0, x_0) \in R^+ \times D$. In view of condition B1 and the inequalities

$$\|I_i^*(x) - I_i(x)\| < \Delta, \qquad x \in D, \qquad i = 1, 2, \dots$$

we establish that $(E + I_i^*): D \to D$. From the last relation and condition A3 it follows that the solution $x_{I^*}(t; \tau_0, x_0)$ exists and does not leave the set D in its definition domain. The quasiuniqueness of the solution follows from condition A1. It remains to show that $x_{I^*}(t; \tau_0, x_0)$ is continuable for any $t \geq \tau_0$. The following two cases are possible:

(i) The integral curve $(t, x_{I^*}(t; \tau_0, x_0))$ for $t > \tau_0$ meets a finite number of hypersurfaces of (4). We shall note the fact that this case is possible only if the domain D is unbounded. Let the meetings be realized at the moments $\tau_1^*, \tau_2^*, \dots, \tau_k^*$. Taking into account that $x_k^{*+} \in D$ as well as condition A3 and the equality

$$x(t; \tau_k^*, x_k^{*+}) = x_{I^*}(t; \tau_0, x_0)$$

which is fulfilled for $t > \tau_k^*$, we conclude that $x_{I^*}(t; \tau_0, x_0) \in D$ for $t > \tau_k^*$. Hence in this case the solution of problem (5), (6), (3) is continuable for any $t > \tau_0$.

(ii) The integral curve $(t, x_{I^*}(t; \tau_0, x_0))$ meets for $t > \tau_0$ infinitely many hypersurfaces of (4). First we shall show that

$$s_i < s_{i+1}, \qquad i = 1, 2, \dots \tag{7}$$

i.e., that the numbers of the hypersurfaces met successively by the integral curve $(t, x_{I^*}(t; \tau_0, x_0))$ are increasing. From conditions B3 and A4 and condition 2 of the lemma we obtain

$$\begin{aligned}
&\tau_i^* - t_{s_i}(x_i^{*+}) \\
&= t_{s_i}(x_i^*) - t_{s_i}(x_i^* + I_{s_i}(x_i^*)) + t_{s_i}(x_i^* + I_{s_i}(x_i^*)) - t_{s_i}(x_i^* + I_{s_i}^*(x_i^*)) \\
&\geq d - \|t_{s_i}(x_i^* + I_{s_i}(x_i^*)) - t_{s_i}(x_i^* + I_{s_i}^*(x_i^*))\| \\
&\geq d - L_{s_i}\|I_{s_i}(x_i^*) - I_{s_i}^*(x_i^*)\| \\
&\geq d - L_{s_i}Md > 0
\end{aligned}$$

From the last inequality and condition A5 we establish that the point $(\tau_i^*, x_i^{*+}) \in \Omega_k$, where $k > s_i$. But then by Lemma 1 the first hypersurface met by the integral curve $(t, x(t; \tau_i^*, x_i^{*+}))$ for $t > \tau_i^*$ is σ_k. The last conclusion means that for $t > \tau_i^*$ the integral curve $(t, x_{I^*}(t; \tau_0, x_0))$ meets first the hypersurface σ_k. Hence $s_{i+1} = k$ and $s_{i+1} > s_i$, i.e., equalities (7) are valid. Then in view of the fact that s_i are integers, we obtain that $s_i \to \infty$ as $i \to \infty$, whence by condition B2 we find that

$$\lim_{i\to\infty} \tau_i^* = \lim_{i\to\infty} t_{s_i}(x_i^*) = \infty$$

From the last relation and the fact that the solution of problem (5), (6), (3) is defined and unique in any of the intervals $(\tau_{i-1}, \tau_i]$, $i = 1, 2, \ldots$, we deduce the assertion of the lemma in this case as well.

Thus Lemma 2 is proved. ■

In the following example we shall show that if condition B3 is violated, then the assertion of Lemma 2 may be not valid.

Example 1. Let $n = 1$ and $D = (-\infty, +\infty)$. Consider the following initial value problem:

$$\frac{dx}{dt} = 0, \quad t \neq \tau_i; \qquad \Delta x(t)|_{t=\tau_i} = I_i(x(\tau_i)), \quad i = 1, 2, \ldots; \qquad x(0) = x_0 \quad (8)$$

where the impulse functions I_i are defined by the equality

$$I_i(x) = I(x) = \begin{cases} 1/x - x, & |x| \geq 1 \\ (|x| - 1)x, & |x| < 1 \end{cases}$$

and the impulse hypersurfaces (in this example the impulse curves) are of the form

$$\sigma_i: \quad t = t_i(x) = \text{arctg}|x| + i\frac{\pi}{2}, \qquad x \in D, \quad i = 1, 2, \ldots$$

Let ε be an arbitrary positive number. Consider the respective perturbed initial value problem

$$\frac{dx}{dt}=0, \quad t\neq\tau_i^*; \qquad \Delta x(t)|_{t=\tau_i^*}=I_i^*(x(\tau_i^*)), \quad i=1,2,\ldots; \qquad x(0)=x_0 \tag{9}$$

where $I_i^*(x)=I_i(x)-\text{sign}(x)\varepsilon$, $x\in D$.

It is trivial to verify that for problem (8) conditions A1–A5, B1, and B2 hold. Moreover, it is clear that

$$\|I_i^*(x)-I_i(x)\|=|I_i^*(x)-I_i(x)|=\varepsilon, \qquad x\in D$$

In spite of this, however, if $|x_0|>\max\{1, 1/\varepsilon\}$, then the solution of problem (9) is not continuable for $t\geq\pi$. In this case the integral curve of (9) meets infinitely many times the hypersurface (curve) σ. This phenomenon is called "beating." This phenomenon is considered in more detail in Dishliev and Bainov (1985).

We stress explicitly that for the impulse curves and the impulse functions of the example we have

$$t_i(x)\geq t_i(x+I_i(x)), \qquad x\in D, \qquad i=1,2,\ldots$$

but condition B3 is not satisfied.

We say that conditions (C) are satisfied if the following conditions hold:

C1. $f(t,0)=0$, $t\in R^+$, $0\in D$.

C2. $I_i(0)=0$, $i=1,2,\ldots$.

From conditions (C) it follows that $x_I(t;\tau_0,0)=0$ for $t\geq\tau_0$.

Definition 2. We say that the zero solution of system (1), (2) is uniformly stable with respect to the impulsive perturbations if $(\forall\varepsilon>0)$ $(\exists\delta=\delta(\varepsilon)>0)$ $(\forall(\tau_0,x_0)\in R^+\times(B_\delta\cap D))$ $(\forall I_i^*: D\to R^n$, $\|I_i^*(x)-I_i(x)\|<\delta$ for $x\in D$, $i=1,2,\ldots)\Rightarrow\|x_{I^*}(t;\tau_0,x_0)\|<\varepsilon$ for $t\geq\tau_0$.

We recall the following definition.

Definition 3. We say that the zero solution of system (1) is uniformly stable if $(\forall\varepsilon>0)$ $(\exists\delta\doteq\delta(\varepsilon)>0)$ $(\forall(\tau_0,x_0)\in R^+\times(B_\delta\cap D))\Rightarrow$ $\|x(t;\tau_0,x_0)\|<\varepsilon$ for $t\geq\tau_0$.

In the above definition, to any $\varepsilon>0$ there correspond infinitely many constants $\delta>0$. Moreover, for any δ corresponding to ε we have $\delta\leq\varepsilon$. Henceforth by $\delta(\varepsilon)$ we shall denote the smallest upper bound of all δ satisfying Definition 2. For each $\varepsilon>0$ we construct the sequence of numbers

$$\delta_1=\delta(\varepsilon), \qquad \delta_2=\delta(\delta_1),\ldots, \qquad \delta_i=\delta(\delta_{i-1}),\ldots$$

The inequalities

$$\delta_i > 0, \qquad \delta_{i+1} \le \delta_i, \qquad i = 1, 2, \ldots$$

hold, hence the sequence $\delta_1, \delta_2, \ldots$ is convergent. Denote by $\delta^* = \delta^*(\varepsilon)$ its limit.

Definition 4. We say that the zero solution of system (1) is strongly uniformly stable if it is uniformly stable and for any $\varepsilon > 0$ the inequality $\delta^*(\varepsilon) > 0$ holds.

Sufficient conditions for strong uniform stability are contained in the following lemma.

Lemma 3. Let the following conditions hold:

1. The zero solution of system (1) is uniformly stable.
2. $(\exists \delta > 0)\ (\forall (\tau_0, x_0) \in R^+ \times (B_\delta \cap D))$ the function $\varphi(t) = \|x(t; \tau_0, x_0)\|$ is monotone decreasing for $t \ge \tau_0$.

Then the zero solution of system (1) is strongly uniformly stable.

In fact, from condition 2 it follows that for any $\varepsilon > 0$ we have $\delta^*(\varepsilon) = \varepsilon > 0$.

In the following example we shall consider an equation for which the equality $\delta(\varepsilon) = \varepsilon$ is not satisfied for any $\varepsilon > 0$. In spite of this, the zero solution of this equation is strongly uniformly stable.

Example 2. In this example by $x(t; \tau_0, x_0)$ we denote the solution of the problem

$$\frac{dx}{dt} = f(t, x), \qquad x(\tau_0) = x_0$$

where

$$f(t, x) = \begin{cases} -2xt, & (t, x) \in R^+ \times [0, 1] \\ -2x[t + (1 - x)K], & (t, x) \in R^+ \times (1, 2) \\ -2x(t - K), & (t, x) \in R^+ \times [2, \infty) \end{cases}$$

K is a positive constant and $(\tau_0, x_0) \in R^+ \times R^+$. The problem considered has a unique solution continuable for any $t \ge \tau_0$. Moreover, for any $x_0 > 0$ we have

$$x(t; \tau_0, x_0) > 0 \qquad \text{for} \quad t \ge \tau_0$$

Then, taking into account that

$$f(t, x) \le -2x(t - K) \qquad \text{for} \quad (t, x) \in R^+ \times R^+$$

we find that

$$0 \leq x(t;\tau_0,x_0) \leq \chi(t;\tau_0,x_0) \tag{10}$$

where $\chi(t;\tau_0,x_0)$ is a solution of the initial value problem

$$\frac{dx}{dt} = -2x(t-K), \qquad x(\tau_0) = x_0$$

It is easy to see that $\chi(t;\tau_0,x_0) = x_0 \exp[(\tau_0 - K)^2 - (t-K)^2]$. Consequently, in view of (10), we conclude that the zero solution of the problem considered is uniformly stable. Moreover, we have:

(i) $\delta(\varepsilon) = \varepsilon$ for $0 < \varepsilon \leq 1$.
(ii) $\varepsilon \exp(-K^2) < \delta(\varepsilon) < \varepsilon$ for $1 < \varepsilon < 2$.
(iii) $\delta(\varepsilon) = \varepsilon \exp(-K^2)$ for $\varepsilon \geq 2$.

From this we conclude that

$$\delta^*(\varepsilon) \geq \varepsilon \exp(-K^2) > 0$$

i.e., that the zero solution of the problem of this example is strongly uniformly stable.

Lemma 4. Let the following conditions be fulfilled:

1. Conditions A1, A3, and C1 hold.
2. The zero solution of system (1) is strongly uniformly stable.

Then $(\forall \tau_0 \in R^+)$ $(\forall \varepsilon > 0)$

$$(\forall x_0 \in B_{\delta^*} \cap D,\ \delta^* = \delta^*(\varepsilon)) \Rightarrow \|x(t;\tau_0,x_0)\| < \delta^* \qquad \text{for} \quad t \geq \tau_0$$

Proof. Let $\varepsilon > 0$, $\delta_1 = \delta(\varepsilon)$, $\delta_2 = \delta(\delta_1), \ldots, \delta^* = \lim_{i\to\infty} \delta_i$, and $(\tau_0, x_0) \in R^+ \times (B_{\delta^*} \cap D)$. Then, since $\|x_0\| < \delta_i$, then

$$\|x(t;\tau_0,x_0)\| < \delta_{i-1} \qquad \text{for} \quad t \geq \tau_0, \quad i = 1, 2, \ldots, \quad \delta_0 = \varepsilon$$

Hence

$$\|x(t;\tau_0,x_0)\| < \delta^* \qquad \text{for} \quad t \geq \tau_0$$

This completes the proof of Lemma 4. ■

Definition 5 (Dannan and Elaydi, 1986). We say that the zero solution of system (1) is uniformly Lipschitz stable if $(\exists G > 0)$ $(\exists g > 0)$ $(\forall (\tau_0, x_0) \in R^+ \times (B_g \cap D)) \Rightarrow \|x(t;\tau_0,x_0)\| < G\|x_0\|$ for $t \geq \tau_0$.

4. MAIN RESULTS

Theorem 1. Let the following conditions be fulfilled:

1. Conditions (A), (B), and (C) hold.

2. The zero solution of system (1) is strongly uniformly stable.

3. $\|x+I_i(x)\|\leq\|x\|/(1+\omega)$, $x\in D$, $i=1, 2, \ldots, \omega>0$.

Then the zero solution of system (1), (2) is uniformly stable with respect to the impulsive perturbations.

Proof. Let $\tau_0\in R^+$ and $\varepsilon>0$, and let $\delta^*=\delta^*(\varepsilon)\leq\varepsilon$, δ^* be the respective constant in the definition of strong uniform stability of the zero solution of system (1). Introduce the notation $\delta=\min(\Delta, M\delta, \omega\delta^*/(1+\omega))$. Let the functions I_i^* satisfy the inequalities

$$\|I_i^*(x)-I_i(x)\|<\delta \qquad \text{for } x\in D, \quad i=1, 2, \ldots$$

Then by Lemma 2 for any point $x_0\in D$ the solution $x_{I^*}(t;\tau_0, x_0)$ is quasi-unique and continuable for $t\geq\tau_0$. Let $\|x_0\|<\delta\leq\delta^*$. From Lemma 4 it follows immediately that

$$\|x_{I^*}(t;\tau_0, x_0)\|=\|x(t;\tau_0, x_0)\|<\delta^*\leq\varepsilon, \qquad \tau_0\leq t\leq\tau_1^*$$

From condition 3 of the theorem and the above estimate for $t=\tau_1^*$ we find

$$\begin{aligned}\|x_1^{*+}\| &= \|x_1^*+I_{s_1}^*(x_1^*)\|\\ &\leq\|x_1^*+I_{s_1}(x_1^*)\|+\|I_{s_1}^*(x_1^*)-I_{s_1}(x_1^*)\|\\ &<\|x_1^*\|/(1+\omega)+\omega\delta^*/(1+\omega)\leq\delta^*\end{aligned}$$

Again by Lemma 4 we obtain

$$\begin{aligned}\|x_{I^*}(t;\tau_0, x_0)\| &= \|x_{I^*}(t;\tau_1^*, x_1^{*+})\|\\ &=\|x(t;\tau_1^*, x_1^{*+})\|\\ &<\delta^*\leq\varepsilon, \qquad \tau_1^*<t\leq\tau_2^*\end{aligned}$$

By induction we obtain the estimates

$$\|x_{I^*}(t;\tau_0, x_0)\|<\delta^*\leq\varepsilon, \qquad \tau_{i-1}^*<t\leq\tau_i^*, \qquad i=1, 2, \ldots$$

This completes the proof of Theorem 1. ■

Theorem 2. Let the following conditions be satisfied:

1. Conditions (A)–(C) hold.

2. The zero solution of system (1) is uniformly Lipschitz stable with a constant G.

3. $\|x+I_i(x)\|\leq\|x\|/(G+\omega)$, $x\in D$, $i=1, \ldots, \omega>0$.

Then the zero solution of system (1), (2) is uniformly stable with respect to the impulsive perturbations.

Proof. Let g be the respective constant in the definition of uniform Lipschitz stability of the zero solution of system (1), $\tau_0\in R^+$, let ε be an arbitrary positive constant, and $\delta = \min(\Delta, Md, g, \varepsilon(G))$. Let the functions I_i^* satisfy the inequalities

$$\|I_i^*(x) - I_i(x)\| < \omega\delta/(G+\omega) \qquad \text{for} \quad x \in D, \quad i = 1, 2, \ldots$$

From Lemma 2 it follows that for any point $x_0 \in D$ the solution $x_{I^*}(t; \tau_0, x_0)$ is quasiunique and continuable for $t \geq \tau_0$. Let $\|x_0\| < \delta \leq g$. Then we obtain

$$\|x_{I^*}(t; \tau_0, x_0)\| = \|x(t; \tau_0, x_0)\| < G\delta \leq \varepsilon, \qquad \tau_0 \leq t \leq \tau_1^*$$

Moreover, we have

$$\begin{aligned} \|x^{*+}\| &= \|x^* + I_{s_1}^*(x^*)\| \\ &\leq \|x^* + I_{s_1}(x^*)\| + \|I_{s_1}^*(x^*) - I_{s_1}(x^*)\| \\ &\leq \|x^*\|/(G+\omega) + \omega\delta/(G+\omega) \leq \delta \leq g \end{aligned}$$

Hence

$$\begin{aligned} \|x_{I^*}(t; \tau_0, x_0)\| &= \|x_{I^*}(t; \tau_1^*, x_1^{*+})\| \\ &= \|x(t; \tau_1^*, x_1^{*+})\| \\ &< G\delta \leq \varepsilon, \qquad \tau_1^* < t \leq \tau_2^* \end{aligned}$$

Finally we find by induction

$$\|x_{I^*}(t; \tau_0, x_0)\| < \varepsilon, \qquad \tau_{i-1}^* < t \leq \tau_i^*, \qquad i = 1, 2, \ldots$$

This completes the proof of Theorem 2. ■

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